Lecture 20

Invariant Subspaces Under a Linear Transformation T and Invariant Direct Sum Decompositions

Definition Let $T \in L(V,V)$ and $U \subset V$ be a subspace. Then U is said to be invariant under T if

$$T(U) \subset U$$

Warning We are not saying that the restriction of T to U is the identity

Examples

Let $V = \mathbb{R}[x]$ and $U = \text{span}(1, x, x^2, x^3) \subset \mathbb{R}[x]$ be the subspace of polynomials of degree less than or equal to 3. Then U is invariant under $T = \frac{d}{dx}$.

However if we take $U = \mathbb{R}[x^2]$ = the even degree polynomial then $\frac{d}{dx} U \underset{\neq}{\subseteq} U$, for example $\frac{d}{dx} x^2 = 2x \notin U$.

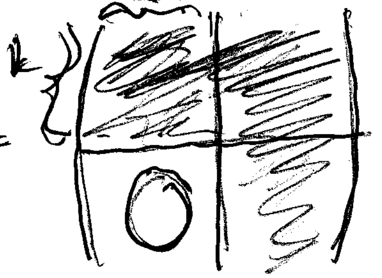
<u>Proposition</u> (proof left to you)

The null space $n(T)$ and the range $T(V)$ of T are both invariant under T

2. Any eigenspace E_λ of T is invariant under T

<u>Important Fact</u>

Suppose the span of the first k standard basis vectors $e_1, e_2, \dots, e_k$ in $\mathbb{R}^n$ is invariant under T. Then A_T, the matrix of T relative to $e_1, e_2, \dots, e_n$, is of the form

$A_T =$ 

Example

$n = 3$, $k = 2$, $U = \text{span}(e_1, e_2)$

$T(e_1) = a e_1 + c e_2 + \underline{0} e_3$ (because U is invariant under T)

$T(e_2) = b e_1 + d e_2 + \underline{0} e_3$ (because U is invariant under T)

$T(e_3) = e e_1 + f e_2 + g e_3$

so

$$A_T = \left(\begin{array}{cc|c} a & b & e \\ c & d & f \\ \hline 0 & 0 & g \end{array}\right)$$

Invariant Direct Sum Decompositions

Suppose $T \in L(V,V)$ and

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_k \qquad (*)$$

Definition

The direct sum decomposition $(*)$ is invariant under T if

$$T(U_1) \subseteq U_1$$
$$\vdots$$
$$T(U_k) \subseteq U_k$$

Definition

A basis $\mathcal{B} = \{b_1, \dots, b_n\}$ is adapted to the direct sum decomposition $(*)$ if $\mathcal{B}$ is the union

$$\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$$

where $\mathcal{B}_1$ is a basis for U_1, $\mathcal{B}_2$ is a basis for U_2, $\dots$, $\mathcal{B}_k$ is a basis for U_k

Remark ((Adapted bases exist))

Given the direct sum decomposition (*), choose a basis $\mathcal{B}_1$ for U_1, then a basis $\mathcal{B}_2$ for U_2, ... and finally a basis $\mathcal{B}_k$ for U_k.

Exercise Put $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$. Then $\mathcal{B}$ is a basis for V.

Lemma Suppose (*) is invariant under $T \in L(V,V)$ and $\mathcal{B}$ is a basis for V adapted to (*). Then the matrix A_T of T relative to $\mathcal{B}$ has block form

$$\left(\begin{array}{c|c|c|c} A_1 & 0 & 0 & 0 \\ \hline 0 & A_2 & 0 & 0 \\ \hline 0 & 0 & A_3 & 0 \\ \hline 0 & 0 & 0 & A_4 \end{array}\right) \qquad \text{(special case } k=4\text{)}$$

Suppose the direct sum decomposition $V = \bigoplus_{i=1}^{k} U_i$ is invariant under $T \in L(V,V)$.

Let $T_i \in L(U_i, U_i)$, $1 \le i \le k$, be the restriction of T to U_i, that is,

$$T_i = T|U_i .$$

We will write $T = \bigoplus_{i=1}^{k} T_i$ and

$$T\left(\sum_{i=1}^{k} u_i\right) = \sum_{i=1}^{k} T(u_i) = \sum_{i=1}^{k} T_i(u_i) .$$

Warning $V = \mathbb{R}^2$, $U = \mathbb{R}e_1$ and $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

The x-axis $\mathbb{R}e_1$ is invariant under T but there is no invariant complement so no T-invariant direct sum decomposition

$$\mathbb{R}^2 = \mathbb{R}e_1 + U_2 \text{, for any } U_2$$

Remark
It very often happens that a T-invariant subspace does not have a T-invariant complement.

The idempotents $\{p_i, 1 \le i \le k\}$ associated to a direct sum decomposition

Definition (see text Definition (23-8), page 196)

Let $p \in L(V,V)$. Then p is said to be an idempotent (latin idem = "the same" potent = power) if

$$p^2 = p$$

(so taking the square of p doesn't change p)

We will now show that commuting idempotents come from and give rise to direct sum decompositions.

Proposition (Proof left to you)

Suppose
$$V = \bigoplus_{i=1}^{k} U_i \qquad (*)$$

Define $p_i \in L(V,V)$, for $1 \le i \le k$, by

$$p_i(u_1 + u_2 + \cdots + u_i + \cdots + u_k) = u_i.$$

Then $p_i^2 = p_i$, $1 \le i \le k$ (1)

and $p_i p_j = p_j p_i$, $1 \le i, j \le k$ (2)

Conversely, given k commuting idempotents $p_1, p_2, \ldots, p_k$ (so (1) and (2) above hold) such ~~that~~

$$I_V = p_1 + p_2 + \cdots + p_k ,$$

put $U_i = p_i(V)$, $1 \le i \le k$.

Then V is the direct sum of the U_i's

$$V = \bigoplus_{i=1}^{k} U_i = \bigoplus_{i=1}^{k} p_i(V) \quad (*)$$

Definition

We will call the idempotents $\{p_i\}$, $1 \le i \le k$, above the idempotents associated to the direct sum decomposition (*)

Problem

Show that $p^2 = p \Rightarrow p^n = p$, $n = 3, 4, 5, \ldots$

Proposition

The direct sum decomposition
$$V = \bigoplus_{i=1}^{k} U_i \quad (*)$$
is invariant under $T \iff$ the associated idempotents $\{p_i\}_{1 \le i \le k}$ commute with T; that is
$$T \circ p_i = p_i \circ T, \quad 1 \le i \le k.$$

Recall p_i is defined by
$$p_i(u_1 + \cdots + u_i + \cdots + u_k) = u_i, \quad 1 \le i \le k$$

Proof $\Leftarrow$ If two linear transformations S and T commute then $S(V)$ and $n(S)$ are invariant under T. So if p_i and T commute for all i then $p_i(V) = U_i$ is invariant under T for all i.

$\Rightarrow$ Let $v \in V$ be given. We are required to prove
$$p_i(T(v)) = T(p_i(v)) \quad (**), \ 1 \le i \le k$$
Write $v = u_1 + u_2 + \cdots + u_k$ with $u_j \in U_j$.

Hence $p_j(v) = u_j$, $1 \le j \le k$,

and in particular

$$p_i(v) = u_i .$$

Hence RHS of $(**)$ $= T(u_i)$

We now show ~~the~~ LHS of $(**)$ is also equal to $T(u_i)$.

First, since $T(U_j) \subset U_j$ we have $T(u_j) \in U_j$ and hence

$$p_i T(u_j) = 0, \text{ all } j \neq i$$

Hence $p_i T(v) = p_i(T(u_i))$

But $T(u_i) \in U_i$ so $p_i T(u_i) = T(u_i)$

as required.

$\square$